Exercise Set 1
Introduction to Poisson processes

1. (A mathematical exercise)

   (a) Let X denote an exponential random variable whose distribution has parameter $\lambda > 0$. Show that $E\,(X) = \lambda^{-1}$ and $var\,(X) = \lambda^{-2}$.

   (b) Let X follow the Poisson law with parameter $\lambda > 0$. Show that $\mathrm{E}(X) = \lambda$.

2. (Additivity of independent Poisson processes) Show that if $\{N_i(t), t \geq 0\}$ are independent Poisson processes with rate λ_i, $i = 1, 2$, then $\{N(t) = N_1(t) + N_2(t), t \geq 0\}$ is a Poisson process with rate $\lambda_1 + \lambda_2$.

3. Let $\{X(t), t \geq 0\}$ be a Poisson process with rate parameter $\lambda = 0.7$, and let T_k be the time of the kth event. Compute:

   (a) $\Pr(X(4) = 5|X(3.5) = 4)$,

   (b) $\Pr(X(4) = 4|X(3.5) = 5)$,

   (c) $\Pr(T_1 < 5)$,

   (d) $\Pr(T_3 < 5|T_2 = 3.5)$,

   (e) $\Pr(X(7) = 8|T_3 = 6)$,

   (f) $\Pr(X(7) = 8|T_9 = 6)$.

4. (Competing risks, queues) Suppose an office receives enquiries from two types of customers: Those who telephone, and those who come in person. Suppose further that the two types of arrival rates are described by independent Poisson process, with rates λ_w for walk-in customers, and λ_r for telephone callers. Denote by T_1 the time of arrival of the first walk-in customer, and denote by N the number of calls in $[0, T_1]$.

   (a) What is the distribution of T_1 ?

(b) What is the probability that the time interval between two consecutive telephone consultations is larger than t?

(c) What is the probability that the first next arriving consultation is a walk-in consultation?

(d) What is the probability that during the time interval $[0, t]$ at most three walk-in consultations arrive?

(e) What is the probability that during the time interval $[0, t]$ at least two consultations arrive?

(f) What is the distribution of the number of calls received before the first walk-in customer arrives ?

    (Further down the line of the argument you might wish to use the fact $\int_0^\infty x^i e^{-x} dx = i!$).

(g) Setting $p = \lambda_w/(\lambda_w + \lambda_r)$ can you give a probabilistic interpretation of the result just derived ?

Optinal mathematical exercises.

1. (optional, a walk through exercise for deriving the law of the Poisson process) Recall that the differential difference equation is, for $j \geq 1$,

$$p_j'(t) = \lambda p_{j-1}(t) - \lambda p_j(t).$$

and, for $j = 0$, $p_0'(t) = -\lambda p_0(t)$. Further note that $N(0) = 0$.

   (a) Show that $p_0(0) = \Pr\{N(0) = 0\} = 1$ and $p_1(0) = 0$.

   (b) Consider $j = 0$. Solve $p_0'(t)/p_0(t) = -\lambda$. Using the boundary condition that $p_0(0) = 1$, show that the solution is

$$p_0(t) = e^{-\lambda t}.$$

   (c) Consider $j = 1$. The equation is now $p_1'(t)+\lambda p_1(t) = \lambda p_0(t) = \lambda e^{-\lambda t}$ with condition $p_1(0) = 0$. Try the solution

$$p_1(t) = \lambda t e^{-\lambda t}.$$

   (d) Consider $j = 2$. The equation is now $p_2'(t) + \lambda p_2(t) = \lambda p_1(t) = \lambda^2 t e^{-\lambda t}$ with condition $p_2(0) = 0$. Try the solution

$$p_2(t) = \frac{1}{2}\lambda^2 t^2 e^{-\lambda t}.$$

2. (optional, a population growth model) A Yule process with parameter $\lambda > 0$ solves the associated differential difference equation equation

$$p_n'(t) = (n-1)\lambda p_{n-1}(t) - n\lambda p_n(t).$$

with $p_0(0) = 1$ and $p_{n>1}(0) = 0$.

   (a) Consider $n = 1$ and show that the solution is

$$p_1(t) = e^{-\lambda t}.$$

   (b) Consider $n = 2$. Find the solution. Hint: Differentiate $e^{2\lambda t}p_2(t)$.