Exercise Set 2
Markov Chains

1. (Properties of P). Show that
   (a) P has non-negative entries, $p_{ij} \geq 0$,
   (b) P has row-sums equal to one, $\sum_j p_{ij} = 1$.

2. Consider a homogeneous Markov chain with state space $\{1, 2, 3\}$ and transition probability matrix $P = [p_{ij}]$ where the transition probabilities are given by
$$p_{ij} = \frac{i+j}{6+3i}$$
   (a) Write out P and check that it is row-stochastic (ie the cells in the row sum to 1).
   (b) Suppose an initial probability distribution is given as $\pi_0 = (1/2, 1/4, 1/4)$. Determine the distribution π_1, i.e. $\Pr(X_1 = i)$ for each state i.

3. Consider the following simple model for the occurence of snow in Paris in February. Let $\{X_0, X_1, ...\}$ be a 2-state Markov chain such that $X_n = 1$ if there is snow in Paris in February of year n, $X_n = 0$ if there is no snow in Paris in February of year n. Two of the transition probabilities are:

   Pr(snow in February of next year l snow in February of this year) = 0.3,

   Pr(no snow in February of next year l no snow in February of this year) = 0.8.

   (a) Write down the transition probability matrix P.
   (b) We have $X_{2018} = 0$, because it has not snowed in Paris this February. What is the probability that the next time we will have snow in Paris in February will be in 2022?
   (c) What is the probability that, given it has not snowed in Paris this February, there will be snow in Paris in February 2020?

4. The transition matrix of a Markov chain with state space $\{0, 1, 2\}$ is given by

$$\begin{pmatrix} .2 & .4 & .4 \\ .2 & .5 & .3 \\ .5 & .2 & .3 \end{pmatrix}$$

Calculate the stationary distribution of this Markov chain.

5. Let $\{X_n\}$ be a homogeneous Markov chain on the state space $S = \{0, 1, ..., N\}$ with transition probability matrix P and assume that for all $j \in S$ there is π_j such that

$$\lim_{n\to\infty} \Pr\{X_n = j\} = \pi_j$$

Express $\lim_{n\to\infty} \Pr\{X_n = 2|X_{n+1} = 1)$ in terms of $(\pi_1, ..., \pi_N)$ and P. Hint: Use the definition of conditional probability.

6. Consider the Markov chain with state space $\{0, 1, 2\}$ and transition probability matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ .1 & .6 & .3 \\ 0 & 0 & 1 \end{pmatrix}$$

   (a) Determine the probability that the Markov chain ends in state 0 given that it starts in state 1. Hint: Let w_i be the probability of absorption into state 0, given that it starts in state i; now consider the conditional probabilities.

   (b) Determine the expected time to absorption, i.e., the expected time for a particle starting in state 1 to enter one of the absorbing states $\{0, 2\}$. Hint: Let D_i be the expected time to absorption, given that it starts in state i; now consider the conditional expectations.

7. Consider the n-step transition probabilities $p_{ij}^{(n)}$, and collect these in the matrix $P_n = [p_{ij}^{(n)}]$.

   (a) Prove the Chapman-Kolmogorov equation

$$p_{ij}^{(m+n)} = \sum_k p_{ik}^{(m)} p_{kj}^{(n)}$$

   i.e. $P_{m+n} = P_m P_n$

   (b) Hence argue that $P_n = P^n$.